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Some properties of conjugate harmonic functions in a half-space

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ABSTRACT

We prove a multi-dimensional analog of the theorem of Hardy and Littlewood about the logarithmic bound of the L^p-average of the conjugate harmonic functions, $0 < p \leqslant 1$. We also give sufficient conditions for a harmonic vector to belong to $H^p(\mathbf{R}^{n+1}_+)$, $0 < p \leqslant 1$.

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1. Introduction and statements of main results

The following result of Hardy and Littlewood [4] is classical.

Theorem 1. *Let $0 < p \leqslant 1$, and let $f(z) = u(z) + iv(z)$ be an analytic function in the unit disc $\mathbf{D} := \{z \in \mathbf{C}\colon |z| < 1\}$, such that*

$$1)\quad v(0) = 0, \qquad 2)\quad M_p(r,u) := \left(\frac{1}{2\pi}\int_0^{2\pi} \left|u\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p} \leqslant C, \quad 0 \leqslant r < 1. \tag{1}$$

Then

$$M_p(r,v) \leqslant AC + AC\left(\log\frac{1}{1-r}\right)^{1/p}. \tag{2}$$

In this paper we prove an analog of Theorem 1 for conjugate harmonic functions in $\mathbf{R}^{n+1}_+ = \mathbf{R}^n \times (0,\infty)$. The case $p < (n-1)/n$ leads to additional difficulties, since $|F|^p$ is subharmonic, provided $p \geqslant (n-1)/n$ [7].

To formulate our first result we briefly introduce some notation and definitions [2]. Let $U(x,y)$ be a harmonic function in $\mathbf{R}^{n+1}_+ \equiv \mathbf{R}^n \times (0,\infty)$. We say that the vector-function $V(x,y) = (V_1(x,y),\dots,V_n(x,y))$ is the conjugate of $U(x,y)$ in the sense of M. Riesz [6,8], if $V_k(x,y)$, $k = 1,\dots,n$ are harmonic functions, satisfying the generalized Cauchy–Riemann conditions:

$$\frac{\partial U}{\partial y} + \sum_{k=1}^{n}\frac{\partial V_k}{\partial x_k} = 0, \qquad \frac{\partial V_i}{\partial x_k} = \frac{\partial V_k}{\partial x_i}, \qquad \frac{\partial U}{\partial x_i} = \frac{\partial V_i}{\partial y}, \quad i \neq k,\ k = 1,\dots,n.$$

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If $U(x,y)$ and $V(x,y)$ are conjugate in $\mathbf{R}_+^{n+1}$ in the above sense, then the vector-function

$$F(x,y)=\big(U(x,y),V(x,y)\big)=\big(U(x,y),V_1(x,y),\dots,V_n(x,y)\big)$$

is called a harmonic vector.

Define

$$M_p(y,F)=\left(\int\limits_{\mathbf{R}^n}\left(U^2(x,y)+\sum_{i=1}^n V_i^2(x,y)\right)^{p/2}dx\right)^{1/p},\quad p>0.$$

The notation $f(x,y)\rightrightarrows^x_{y\to\infty}0$ means that $f(x,y)$ converges to 0 uniformly with respect to x, provided $y\to\infty$.

We have

Theorem 2. *Let $0<p\leqslant 1$, and let $F=(U,V_1,\dots,V_n)$ be the harmonic vector such that*

$$1)\quad V_i\rightrightarrows^x_{y\to\infty}0,\quad i=1,\dots,n,\qquad 2)\quad M_p(y,U)\leqslant C,\qquad 3)\quad M_p(1,F)\leqslant C. \tag{3}$$

Then

$$M_p(y,V)\leqslant AC+AC|\log y|^{1/p}. \tag{4}$$

The third condition in (3) appears after the application of the Main Theorem of Calculus, see (8), (9). The logarithmic bound comes from the estimate

$$\int\limits_{\mathbf{R}^n}\Big(\sup_{\xi\geqslant t}\big|\nabla U(x,\xi)\big|\Big)^p dx\leqslant ACt^{-p}$$

in the integral

$$\int\limits_y^1 t^{p-1}\,dt\int\limits_{\mathbf{R}^n}\Big(\sup_{\xi\geqslant t}\big|\nabla U(x,\xi)\big|\Big)^p dx,$$

see Lemmata 3, 4, 5.

To control the logarithmic blow up, we use the “Littlewood–Paley”-type condition:

$$I(p):=\int\limits_0^1 t^{p-1}\,dt\int\limits_{\mathbf{R}^n}\Big(\sup_{\xi\geqslant t}\big|\nabla U(x,\xi)\big|\Big)^p dx<\infty.$$

Our second result is

Theorem 3. *Let $0<p\leqslant 1$, and let $F=(U,V_1,\dots,V_n)$ be the harmonic vector such that*

$$1)\quad F\rightrightarrows^x_{y\to\infty}0,\qquad 2)\quad M_p(1,F)\leqslant C,\qquad 3)\quad I(p)<\infty. \tag{5}$$

Then $F\in H^p$.

The methods of the proof are similar to those in [5]. In particular, we use Lemma 2 [5] to prove Theorem 2 (see the proof of Lemma 5). Nevertheless, the results obtained in this note do not follow from [5] and represent an independent interest.

The paper is organized as follows. In Section 2 we give all necessary definitions and auxiliary results used in the sequel. In Sections 3 and 4 we prove Theorems 2 and 3. For convenience of the reader we split our proofs into elementary lemmata.

2. Auxiliary results

To define the space $H^p(\mathbf{R}_+^{n+1})$ we follow the work of Fefferman and Stein [2]. Let $U(x,y)$ be a harmonic function in $\mathbf{R}_+^{n+1}$, and let $U_{j_1j_2j_3\dots j_k}$ denote a component of a symmetric tensor of rank k, $0\leqslant j_i\leqslant n$, $i=1,\dots,n$. Suppose also that the trace of our tensor is zero, meaning

$$\sum_{j=0}^n U_{jjj_3\dots j_k}(x,y)=0,\quad \forall j_3,\dots,j_k.$$

The tensor of rank $k+1$ can be obtained from the above tensor of rank k by passing to its gradient:

$$U_{j_1 j_2 \dots j_k j_{k+1}}(x, y) = \frac{\partial}{\partial x_{j_{k+1}}} \big(U_{j_1 j_2 j_3 \dots j_k}(x, y)\big), \quad x_0 = y,\ 0 \leqslant j_{k+1} \leqslant n.$$

Definition. (See [2].) We say that $U \in H^p(\mathbf{R}^{n+1}_+)$, $p > 0$, if there exists a tensor of rank k of the above type with the properties:

$$U_{0\dots0}(x, y) = U(x, y), \qquad \sup_{y>0} \int_{\mathbf{R}^n} \bigg(\sum_{(j)} U^2_{(j)}(x, y) \bigg)^{p/2} dx < \infty, \quad (j) = (j_1, \dots, j_k).$$

It is well known that the function $(\sum_{(j)} U^2_{(j)}(x, y))^{p/2}$ is subharmonic for $p \geqslant p_k = (n-k)/(n+k-1)$, see [1,2,8].

We remind that the *radial* and the *non-tangential* maximal functions are defined as follows:

$$F^+(x) = \sup_{y>0} \big|F(x, y)\big|, \qquad N_\alpha(F)\big(x^0\big) = \sup_{(x,y)\in\Gamma_\alpha(x^0)} \big|F(x, y)\big|.$$

Here

$$\Gamma_\alpha\big(x^0\big) = \big\{(x, y) \in \mathbf{R}^{n+1}_+:\ \big|x - x^0\big| < \alpha y\big\}, \quad \alpha > 0,$$

is an infinite cone with the vertex at x^0. It is well known [2] that

$$F \in H^p\big(\mathbf{R}^{n+1}_+\big) \quad \Longleftrightarrow \quad N_\alpha(F) \in L^p \quad \Longleftrightarrow \quad F^+ \in L^p, \quad p > 0.$$

We also define the *weak maximal function*

$$WF(x, y) = \sup_{\zeta \geqslant y} \big|F(x, \zeta)\big|, \quad y > 0.$$

The above expression is understood as follows: we fix x, and for fixed y we find the supremum over all $\zeta \geqslant y$.

We will use the following results.

Lemma 1. *(See [2, p. 173].) Suppose w is harmonic in $\mathbf{R}^{n+1}_+$, and $M_p(y, u) \leqslant C$ for some p, $0 < p < \infty$. Then*

$$\sup_{x\in\mathbf{R}^n} \big|u(x, y)\big| \leqslant A y^{-n/p}, \quad 0 < y < \infty. \tag{6}$$

Theorem 4. *(See [3, p. 267].) Let $0 < p \leqslant 1$, $a > 0$, let $w : \mathbf{R}^{n+1}_+ \to [0, \infty)$ be a function such that w^p is subharmonic and satisfies*

$$J_{a,p} := \int_{\mathbf{R}^{n+1}_+} t^{ap-1} w(x, t)^p \, dx\, dt < +\infty,$$

and for each $(x, t) \in \mathbf{R}^n \times [0, +\infty)$ let

$$w_a(x, t) := \frac{1}{\Gamma(a)} \int_0^{+\infty} s^{a-1} w(x, s + t)\, ds.$$

Then w_a is subharmonic on $\mathbf{R}^{n+1}_+$ and is finite a.e. on $\mathbf{R}^n$, and for all $t \geqslant 0$,

$$\int_{\mathbf{R}^n} w(x, t)^p \, dx \leqslant AC(a, n, p) J_{a,p}.$$

Theorem 5. *(See [3, p. 269].) Let $m \in \mathbf{N}$, $p \geqslant (n-1)/(m+n-1)$ (if $n = 1$ we suppose $p > 0$), and let $u : \mathbf{R}^{n+1}_+ \to \mathbf{R}$ be harmonic. Then, for all $t > 0$,*

$$\int_{\mathbf{R}^n} \big|\nabla^m u(x, t)\big|^p \, dx \leqslant A(m, n, p) t^{-mp-1} \int_{t/2}^{3t/2} ds \int_{\mathbf{R}^n} \big|u(x, s)\big|^p \, dx.$$

Lemma 2. *(See [5, p. 2464].) Let $p > 0$ and let $F = (U, V_1, \dots, V_n)$ be such that $V_i \rightrightarrows^{x}_{y\to\infty} 0$, $i = 1, \dots, n$, $M_p(y, U) \leqslant C$. Then*

$$M_p\big(y, \nabla^k F\big) \leqslant ACy^{-k}, \quad k \in \mathbf{N}.$$

Notation. We denote by $D_i^k f(x,y)$ the partial derivative of the function f of the order k with respect to x_i, $i=1,2,\dots,n+1$, $\nabla^k f(x)$ stands for $(\frac{\partial^k f(x)}{\partial x_1^k},\dots,\frac{\partial^k f(x)}{\partial x_n^k})$. Everywhere below the constants $A(k,n), C, K$ depend only on the parameters pointed in parentheses, and may be different from line to line.

3. Proof of Theorem 2

Lemma 3. *Let $0<p\leqslant 1$, and let $F=(U,V_1,\dots,V_n)$ satisfy $M_p(1,F)\leqslant C$. Then*

$$M_p(y,V_i)\leqslant AC+\int_{\mathbf{R}^n}\left(\int_y^1 \sup_{\xi\geqslant t}\left|\nabla U(x,\xi)\right|dt\right)^p dx,\quad i=0,1,\dots,n,\ V_0=U. \tag{7}$$

Proof. By the Main Theorem of Calculus, and the Cauchy–Riemann equations, we have

$$V_i(x,y)-V_i(x,1)=-\int_y^1\frac{\partial V_i(x,t)}{\partial t}\,dt=-\int_y^1\frac{\partial U(x,t)}{\partial x_i}\,dt, \tag{8}$$

$i=1,2,\dots,n$,

$$U(x,y)-U(x,1)=-\int_y^1\frac{\partial U(x,t)}{\partial t}\,dt. \tag{9}$$

Then,

$$M_p(y,V_i)\leqslant M_p(1,V_i)+\int_{\mathbf{R}^n}\left(\int_y^1\sup_{\xi\geqslant t}\left|\nabla U(x,\xi)\right|dt\right)^p dx,$$

and the result follows. $\square$

Lemma 4. *Let $0<p\leqslant 1$ and let $0<y<1$. Then*

$$\int_{\mathbf{R}^n}\left(\int_y^1\sup_{\xi\geqslant t}\left|\nabla U(x,\xi)\right|dt\right)^p dx\leqslant y^p\int_{\mathbf{R}^n}\Bigl(\sup_{\xi\geqslant y}\left|\nabla U(x,\xi)\right|\Bigr)^p dx+2p\int_y^1 t^{p-1}\,dt\int_{\mathbf{R}^n}\Bigl(\sup_{\xi\geqslant t}\left|\nabla U(x,\xi)\right|\Bigr)^p dx. \tag{10}$$

Proof. Denote

$$\Psi(x,y):=\int_y^1\sup_{\xi\geqslant t}\left|\nabla U(x,\xi)\right|dt.$$

Following [4] consider

$$\Phi(x,y):=\Psi(x,y)^p-y^p\left(-\frac{\partial\Psi(x,y)}{\partial y}\right)^p,\qquad \Omega(y):=\bigl\{x\in\mathbf{R}^n:\ \Phi(x,y)>0\bigr\}.$$

By definition of $\Omega(y)$,

$$\int_{\mathbf{R}^n\setminus\Omega(y)}\Psi(x,y)^p\,dx\leqslant y^p\int_{\mathbf{R}^n\setminus\Omega(y)}\left(-\frac{\partial\Psi(x,y)}{\partial y}\right)^p dx. \tag{11}$$

Next, the reasons which are similar to those in [4], imply

$$\int_{\Omega(y)}\Phi(x,y)\,dx-\int_{\Omega(a)}\Phi(x,a)\,dx=\int_y^a d\xi\int_{\Omega(\xi)}-\frac{\partial\Phi(x,\xi)}{\partial\xi}\,dx,\quad 0<y<a\leqslant 1. \tag{12}$$

Moreover, using $\partial^2\Psi/\partial\xi^2\geqslant 0$ for almost every $0<\xi<1$, we have

$$-\frac{\partial \Phi}{\partial \xi} = -p\Psi^{p-1}\frac{\partial \Psi}{\partial \xi} + p\xi^{p-1}\left(-\frac{\partial \Psi}{\partial \xi}\right)^{p} - p\xi^{p}\left(-\frac{\partial \Psi}{\partial \xi}\right)^{p-1}\frac{\partial^2 \Psi}{\partial \xi^2}$$
$$\leqslant p\left(-\frac{\partial \Psi}{\partial \xi}\right)\left(\Psi^{p-1} + \xi^{p-1}\left(-\frac{\partial \Psi}{\partial \xi}\right)^{p-1}\right) \leqslant 2p\xi^{p-1}\left(-\frac{\partial \Psi}{\partial \xi}\right)^{p}.$$

Here the last inequality follows from the definition of $\Omega(\xi)$. Since

$$\Psi(x, y) \leqslant (1-y)\left|\frac{\partial \Psi(x, y)}{\partial y}\right|,$$

the function $\Phi(x, y)$ is negative, provided y is sufficiently close to 1, and we can take a such that $\Omega(a) = \emptyset$. Hence, (12) yields

$$\int_{\Omega(y)} \Phi(x, y)\,dx \leqslant 2p\int_{y}^{a} \xi^{p-1}\,d\xi \int_{\Omega(\xi)} \left(-\frac{\partial \Psi(x, \xi)}{\partial \xi}\right)^{p} dx \leqslant 2p\int_{y}^{1} \xi^{p-1}\,d\xi \int_{\mathbf{R}^n} \left(-\frac{\partial \Psi(x, \xi)}{\partial \xi}\right)^{p} dx. \tag{13}$$

Adding

$$\int_{\Omega(y)} \Psi(x, y)^p\,dx \leqslant \int_{\Omega(y)} \Phi(x, y)\,dx + y^p \int_{\Omega(y)} \left(-\frac{\partial \Psi(x, y)}{\partial y}\right)^{p} dx$$

with (11), and using (13), we obtain (10). □

The next result is crucial.

Lemma 5. *Let $p > 0$ and let $F = (U, V_1, \dots, V_n)$ be such that*

$$1)\quad V_i \rightrightarrows^{x}_{y\to\infty} 0, \quad i = 1, \dots, n, \qquad 2)\quad M_p(y, U) \leqslant C. \tag{14}$$

Then

$$\left(\int_{\mathbf{R}^n} \left(\sup_{\xi \geqslant y}\left|\phi_{ij}(x, \xi)\right|\right)^{p} dx\right)^{1/p} \leqslant ACy^{-1}, \tag{15}$$

where $\phi_{ij}(x, y)$ is a coordinate of $\nabla V_i(x, y)$, $j = 1, \dots, n+1$, $x_{n+1} = y$, $i = 0, \dots, n$, $V_0 = U$.

Proof. Fix $p > 0$ and let $l = \inf\{j \in \mathbf{N}\colon\ p > p_j := (n-1)/(j+n-1)\}$. Since $\nabla V_i(x, y) \rightrightarrows^{x}_{y\to\infty} 0$, we may use the following relation (see [3] or [2])

$$\phi_{ij}(x, y) = \frac{1}{(2l-2)!}\int_{y}^{\infty} (s-y)^{2l-2} D^{2l-1}_{n+1}\phi_{ij}(x, s)\,ds = \frac{1}{(2l-2)!}\int_{0}^{\infty} s^{2l-2} D^{2l-1}_{n+1}\phi_{ij}(x, s+y)\,ds.$$

We have

$$\sup_{\xi \geqslant y}\left|\phi_{ij}(x, \xi)\right| \leqslant R(x, y),$$

where

$$R(x, y) := \frac{1}{(2l-2)!}\int_{0}^{\infty} s^{2l-2}\left(\sup_{\xi \geqslant y}\left|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x, s+\xi)\right|\right) ds.$$

To prove (15) it is enough to show that

$$M_p(y, R) \leqslant ACy^{-1}. \tag{16}$$

Since

$$\left(\left|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x, \xi)\right|\right)^p$$

is subharmonic [1], the function

$$w^p(x, s+y) := \left(\sup_{\xi \geqslant y}\left|\nabla^l D^{l-1}_{n+1}\phi_{ij}(x, s+\xi)\right|\right)^p$$

is also subharmonic, and we may apply Theorem 4 (take $a = 2l-1$, $A = A(l,n,p)$) to obtain

$$\int_{\mathbf{R}^n} \left|R(x,y)\right|^p dx \leqslant A \int_0^\infty s^{(2l-1)p-1}\, ds \int_{\mathbf{R}^n} \Big(\sup_{\xi \geqslant y} \left|\nabla^l D_{n+1}^{l-1} \phi_{ij}(x, s+\xi)\right|\Big)^p dx$$
$$= A \int_0^\infty s^{(2l-1)p-1}\, ds \int_{\mathbf{R}^n} \Big(\sup_{\xi \geqslant y} \left|\nabla^l D_{n+1}^{l-1} \phi_{ij}(x, s+\xi)\right|^{p_j}\Big)^{p/p_j} dx.$$

By the choice of p_j, we have $p/p_j > 1$ and we may use the well-known [2] L^{p/p_j}-boundedness of the maximal operator:

$$\int_{\mathbf{R}^n} \left|R(x,y)\right|^p dx \leqslant A \int_0^\infty s^{(2l-1)p-1}\, ds \int_{\mathbf{R}^n} \left|\nabla^l D_{n+1}^{l-1} \phi_{ij}(x, s+y)\right|^p dx.$$

Since $D_{n+1}^{l-1} \phi_{ij}(x,y)$ is the l-th derivative of V_i, we use Lemma 2 to get

$$\int_{\mathbf{R}^n} \left|\nabla^l D_{n+1}^{l-1} \phi_{ij}(x,y)\right|^p dx \leqslant \int_{\mathbf{R}^n} \left|\nabla^{2l} F(x,y)\right|^p dx \leqslant C y^{-2lp}. \tag{17}$$

This gives

$$\int_{\mathbf{R}^n} \left|R(x,y)\right|^p dx \leqslant A(l,n,p) C \int_0^\infty s^{(2l-1)p-1} (s+y)^{-2lp}\, ds = A(l,n,p) C y^{-p},$$

and (16) is proved. □

Proof of Theorem 2. The proof follows from Lemmata 3, 4, 5. □

4. Proof of Theorem 3

Lemma 6. *Let $p > 0$. Then $F = (U, V_1, \ldots, V_n) \in H^p$ iff*

$$1)\quad F \rightrightarrows^{x}_{y\to\infty} 0, \qquad 2)\quad \int_{\mathbf{R}^n} \Big(\sup_{\eta \geqslant y} \left|U(x,\eta)\right|\Big)^p dx < C.$$

Proof. Let $F \in H^p$, then both 1) and 2) are well known [2]. We prove the converse in two steps. At first we show that

$$\int_{\mathbf{R}^n} \Big(\sup_{y>0} \left|U(x,y)\right|\Big)^p dx \leqslant C. \tag{18}$$

Then we prove that (18) implies

$$\Big(\sup_{y>0} \left|V_i(\cdot, y)\right|\Big)^p \in L^1\big(\mathbf{R}^n\big), \quad i = 1, \ldots, n. \tag{19}$$

To prove (18), we observe that

$$\sup_{y>0} \left|U(x,y)\right| = \sup_{y>0} \sup_{\eta \geqslant y} \left|U(x,\eta)\right| = \lim_{y\to 0} \sup_{\eta \geqslant y} \left|U(x,\eta)\right|.$$

Hence, using 2) and Fatou's Lemma, we obtain

$$\int_{\mathbf{R}^n} \Big(\sup_{y>0} \left|U(x,y)\right|\Big)^p dx \leqslant \lim_{y\to 0} \int_{\mathbf{R}^n} \Big(\sup_{\eta \geqslant y} \left|U(x,\eta)\right|\Big)^p dx \leqslant C.$$

It remains to show (19). Using Cauchy–Riemann equations, we have

$$V_i(x,y) = [V_i]_{0\ldots 0}(x,y) = \frac{(-1)^k}{(k-1)!} \int_0^\infty s^{k-1} D_{n+1}^k V_i(x, s+y)\, ds = \frac{(-1)^k}{(k-1)!} \int_0^\infty s^{k-1} D_{n+1}^{k-1} D_i U(x, s+y)\, ds, \tag{20}$$

where k is chosen such that the function $(\sum_{(j)} U^2_{(j)}(x,y))^{p/2}$ is subharmonic ($p \geqslant p_k = (n-k)/(n+k-1)$, see [1,2,8]). Since the expression in (20) is one of the tensor coordinates of $U_{(j)}$, see [2, p. 169], we have

$$\int_{\mathbf{R}^n} \Big(\sup_{y>0} \big|V_i(x,y)\big|\Big)^p\, dx \leqslant \int_{\mathbf{R}^n} \Big(\sup_{y>0} \sum_{(j)} U^2_{(j)}(x,y)\Big)^{p/2} dx$$

$$= \sup_{y>0} \int_{\mathbf{R}^n} \Big(\sum_{(j)} U^2_{(j)}(x,y)\Big)^{p/2} dx \leqslant C \int_{\mathbf{R}^n} \Big(\sup_{y>0} \big|U(x,y)\big|\Big)^p\, dx.$$

The last estimate is proved in [2, p. 170]. □

Lemma 7. *Let $p > 0$ and let $F = (U, V_1, \dots, V_n)$ be such that $V_i \rightrightarrows^{x}_{y\to\infty} 0$, $i = 1, \dots, n$. Then $F \in H^p$, provided*

$$1)\quad M_p(1,F) \leqslant C, \qquad 2)\quad M_p(y,U) \leqslant C, \qquad 3)\quad I(p) < \infty. \tag{21}$$

Proof. At first we prove that

$$2^{-p} \int_G \Big(\sup_{y>0} \big|V_i(x,y)\big|\Big)^p\, dx \leqslant AC + \int_G \Big(\sup_{y>0} \big|\nabla U(x,y)\big|\Big)^p\, dx, \quad \forall G \subset \mathbf{R}^n. \tag{22}$$

This follows from (8), (9), the inequality

$$2^{-p}\Big(\sup_{y>0} \big|V_i(x,y)\big|\Big)^p \leqslant \big(\big|V_i(x,1)\big|\big)^p + \Big(\sup_{y>0} \big|\nabla U(x,y)\big|\Big)^p,$$

$j = 0, 1, 2, \dots, n$, $V_0 = U$, and the first condition in (21).

Now we finish the proof. Assume that the lemma is not true. Then $\forall N > 0$ there exists a set $E \subset \mathbf{R}^n$, $0 < m(E) < \infty$, such that

$$\int_E \Big(\sup_{y>0} \big|V_i(x,y)\big|\Big)^p\, dx \geqslant 4N,$$

for some $i = 0, 1, \dots, n$, $V_0 = U$. Hence, taking $G = E$ in (22) we have

$$\int_E \Big(\sup_{y>0} \big|\nabla U(x,y)\big|\Big)^p\, dx \geqslant 2N.$$

Then

$$\sup_{y>0} \big|\nabla U(x,y)\big| = \sup_{y>0} \sup_{\xi\geqslant y} \big|\nabla U(x,\xi)\big| = \lim_{y\to 0} \sup_{\xi\geqslant y} \big|\nabla U(x,\xi)\big|$$

and Fatou's Lemma imply

$$\varliminf_{y\to 0} \int_E \Big(\sup_{\xi\geqslant y} \big|\nabla U(x,\xi)\big|\Big)^p\, dx \geqslant \int_E \Big(\sup_{y>0} \big|\nabla U(x,y)\big|\Big)^p\, dx \geqslant 2N.$$

But this contradicts the third condition of the lemma, since

$$I(p) \geqslant \int_0^1 dy \int_{\mathbf{R}^n} \Big(\sup_{\xi\geqslant y} \big|\nabla U(x,\xi)\big|\Big)^p\, dx \geqslant \int_0^1 dy \int_E \Big(\sup_{\xi\geqslant y} \big|\nabla U(x,\xi)\big|\Big)^p\, dx \geqslant N. \qquad \square$$

Lemma 8. *Let $p > 0$ and let $F = (U, V_1, \dots, V_n)$ be the harmonic vector satisfying conditions of Theorem* 3. *Then* (18) *holds.*

Proof. We argue as in the previous lemma. We have (9), and

$$2^{-p} \int_{\mathbf{R}^n} \Big(\sup_{y>0} \big|U(x,y)\big|\Big)^p\, dx \leqslant \int_{\mathbf{R}^n} \big|U(x,1)\big|^p\, dx + \int_{\mathbf{R}^n} \Big(\sup_{y>0} \big|\nabla U(x,y)\big|\Big)^p\, dx$$

leads to a contradiction. □

Proof of Theorem 3. The proof follows from Lemmata 8, 7 and 6. □

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